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AN OPTIMAL QUASI SOLUTION FOR THE CAUCHY PROBLEM FOR LAPLACE EQUATION IN THE FRAMEWORK OF INVERSE ECG

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Abstract. The inverse ECG problem is set as a boundary data completion for the Laplace equation: at each time the potential is measured on the torso and its normal derivative is null. One aims at reconstructing the potential on the heart. A new regularization scheme is applied to obtain an optimal regularization strategy for the boundary data completion problem. We consider the $\mathbb{R}^{n+1}$ domain Ω. The piecewise regular boundary of Ω is defined as the union $\partial\Omega = \Gamma_1 \cup \Gamma_0 \cup \Sigma$, where Γ_1 and Γ_0 are disjoint, regular, and n-dimensional surfaces. Cauchy boundary data is given in Γ_0, and null Dirichlet data in Σ, while no data is given in Γ_1. This scheme is based on two concepts: admissible output data for an ill-posed inverse problem, and the conditionally well-posed approach of an inverse problem. An admissible data is the Cauchy data in Γ_0 corresponding to an harmonic function in $C^2(\Omega) \cap \mathrm{H}^1(\Omega)$. The methodology roughly consists of first characterizing the admissible Cauchy data, then finding the minimum distance projection in the L^2-norm from the measured Cauchy data to the subset of admissible data characterized by given *a priori* information, and finally solving the Cauchy problem with the aforementioned projection instead of the original measurement.

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1. Introduction

For a century, the main tool for cardiologists to assess the electric wave triggering the contraction of a patient's heart has been the ECG. It provides the effect of the heart potential on the potential at various points of the body surface with a good time resolution. Cardiologists are trained to interpret it, but the resulting information is qualitative. It is thought that building a potential map on the heart from the measurement of potential on the torso will allow a major improvement for the diagnosis. Such devices already exist but the stage of solving numerically a Cauchy problem for the Laplace equation remains the key point to improve.

Given a domain Ω and two separate parts Γ_0 and Γ_1 of its boundary $\partial\Omega$, this problem, understood as the boundary value completion on Γ_1, from given Cauchy data on Γ_0, of an harmonic potential defined over Ω, has been widely studied because of its various medical and engineering applications [2, 6, 8, 13]. Many examples in the literature show that this Cauchy problem can be approached as a linear inverse problem, severely ill-posed

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in the Hadamard sense [7,17]. Improving accuracy of the regularized solution still is a relevant issue [10], even though there are regularization strategies of optimal order and asymptotically optimal for this kind of problem in the framework of general theory [14,22,23,25]. Throughout these pages, a new methodology (or scheme) is applied to get an optimal approximation to the solution of the previous boundary data completion problem, referred to as the Admissible Data Methodology (**AD**). The Cauchy problem for Laplace equation that will be considered is as follows:

$$\begin{aligned} \Delta u &\equiv 0 \;\; in\ \Omega, && (1)\\ u|_{\Sigma} &= 0, && (2)\\ u|_{\Gamma_0} &= \phi, && (3)\\ \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_0} &= \psi, && (4) \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, ν is the unit outward normal to $\partial\Omega$, $\frac{\partial u}{\partial \nu}|_{\Gamma_0}$ the normal derivative of u over Γ_0, and $\partial\Omega$ is composed of the smooth, and open surfaces Γ_1, Γ_0 and $\Sigma = \partial\Omega \setminus (\Gamma_0 \cup \Gamma_1)$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$, Σ a surface of strictly positive measure in $\partial\Omega$. For us, problem (1)-(4) will be the theoretical model describing some real phenomenon. The model clearly requires that the pair (ϕ, ψ) belong to a certain class M of functions where problem (1)-(4) has a solution, and it makes sense to extend that solution over Γ_1 in the same sense in which boundary conditions over Γ_0 are given.

The methodology described in following pages provides a regularization strategy based on a *model testing* idea: if for model (1)-(4) a given measured Cauchy data $(\tilde{\phi}, \tilde{\psi})$ is in a function class containing M, with noise level $\delta > 0$, it must exist an element of M which is close enough to $(\tilde{\phi}, \tilde{\psi})$ (of order δ); otherwise, there would be evidence pointing that the model does not describe the phenomenon properly. Based on this thought, a regularization scheme consists in regularizing the ill-posed problem of minimizing the distance from the measurement to the *admissible data set* M. For that purpose we find a subclass of M where projection is conditionally well posed, and we use that projection to solve problem (1)-(4).

In practice, the solution of the boundary data completion problem using the projection as the new Cauchy data must approximate well enough the solution of the real boundary data completion problem for real phenomenon, whether no evidence to reject the model is observed. This is a theoretical work, and so only the regularization strategy of the boundary data completion problem concerning the Cauchy problem (1)-(4) is considered, no statistical approach toward a hypothesis test is considered.

In the sense of this modeling philosophy, an *exact data* is understood as the best theoretical modeling data; this way, $(\phi^\dagger, \psi^\dagger)$ will denote the exact Cauchy data, and $\Phi^\dagger$ the exact solution of the boundary data completion related to (1)-(4) and $(\phi^\dagger, \psi^\dagger)$. Solution of (1)-(4) will be considered in a weak sense and recovering Dirichlet boundary data means finding a function Φ in the Sobolev space $\mathrm{H}^{\frac{1}{2}}(\Gamma_1)$, and a harmonic function u in $\mathrm{H}^1(\Omega)$ that satisfy (2)-(4) and

$$u|_{\Gamma_1} = \Phi, \tag{5}$$

where Dirichlet data ϕ belongs to $\mathrm{H}^{\frac{1}{2}}(\Gamma_0)$ and Neumann boundary condition ψ is taken in $\mathrm{L}^2(\Gamma_0)$ as *a priori* information, although it is usually considered in the Sobolev space $\mathrm{H}^{-\frac{1}{2}}(\Gamma_0)$. Numerical examples are given in section 4 when Ω is a cylindrical domain.

A common way to set boundary completion problem is via the operational equation

$$A\Phi = \rho, \quad \text{where the right hand side is defined by} \quad \rho \equiv \phi - B\psi,$$

A being the Dirichlet to Dirichlet operator from Γ_1 to Γ_0 defined by auxiliary boundary problem given by (1), (2), (5) and a null Neumann condition over Γ_0, and B is the Neumann to Dirichlet operator on Γ_0 defined by the auxiliary boundary problem given by (1), (2), (4) and a null Dirichlet condition on Γ_1.

When both components of Cauchy data are given with noise of order δ at most, then, the continuous dependency of ρ allows to set the inverse problem as a first kind ill-posed one:

$$A\Phi^\dagger = \rho^\dagger, \quad \rho^\dagger = \phi^\dagger - B\psi^\dagger \quad \left\|\phi^\dagger - \tilde{\phi}\right\| \leq \delta \quad \left\|\psi^\dagger - \tilde{\psi}\right\| \leq \delta,$$

$\Phi^\dagger$ being the exact solution for the unknown exact Cauchy data $(\phi^\dagger, \psi^\dagger)$, and $(\tilde{\phi}, \tilde{\psi})$ the known perturbation of $(\phi^\dagger, \psi^\dagger)$. In this approach, all methods use *a priori* information (some source condition or noise level) to regularize the Moore-Penrose pseudo-inverse of A i.e. to look for the Φ^{reg} in the A domain minimizing the discrepancy $\|A\Phi - \tilde{\rho}\|$. Hence, regularization strategies such as Tikhonov, Landweber, Truncated Singular Value Decomposition (TSVD), conjugate gradient or discretization methods are applied. All these strategies provide order optimal or asymptotically optimal solutions under the *a priori* information that the exact solution $\Phi^\dagger$ is bounded by a given constant in a stronger norm than the one of $\mathrm{L}^2(\Gamma_1)$; however, they require a high smoothness condition over the exact solution in order to obtain optimal approximations in the case of severely ill-posed problems [1].

What we do here is to focus on the *admissible data*, which will be understood as the Cauchy data such that the weak solution of problem (1)-(4) exists. Set of admissible data is characterized via the operational equation defining the inverse problem, and given *a priori* information is used to continuously project in some sense the perturbed data over a subset of admissible data and in a such way that the inverse problem is conditionally well-posed over the aforementioned set. Our regularization scheme consists of two simple steps approximately described as follows:

(1) Finding $(\phi^\delta, \psi^\delta)$, the minimum L^2-distance projection from the perturbed Cauchy data $(\tilde{\phi}, \tilde{\psi})$ over the subset of admissible data defined by given *a priori* information such that the worst case error defined in section 1.3 of [1] is bounded (the conditionally well-posed problem).
(2) Solving the inverse problem with admissible data $(\phi^\delta, \psi^\delta)$ instead of the first given Cauchy data $(\tilde{\phi}, \tilde{\psi})$.

In this work the presented methodology will be called the admissible data methodology (AD), and its optimality will be proved as well. In section 3 it is proved that the AD solution is a quasi-solution according with definition in [4], and that it does not require a so strong *a priori* information to be optimal, as occurs with Tikhonov regularization strategy, the most popular one, in the framework of the severely ill-posed problems [1].

Through this scheme, considering a stronger *a priori* information than before, and applying the classic optimization theory, we show that in such cases there is $\alpha \geq 0$, depending on the stronger *a priori* information, such that Φ^α defined by the following optimization problem is an optimal approximation of the exact solution for the inverse problem,

$$(\Phi^\alpha, \psi^\alpha) \quad = \quad \underset{(\Phi,\psi)}{arg\ min} \quad \left\|A\Phi + B\psi - \tilde{\phi}\right\|^2 + \left\|\psi - \tilde{\psi}\right\|^2 + \alpha \left\|(\Phi, \psi)\right\|_0^2$$

where A is the linear compact operator defining the inverse problem, B the Neumann to Dirichlet operator on Γ_0 for the auxiliary boundary data problem defined by (1)-(2) and (4)-(5) with $\Phi \equiv 0$, and $\|(\cdot,\cdot)\|_0$ is a stronger norm than the $\mathrm{H}^{\frac{1}{2}}(\Gamma_1) \times \mathrm{L}^2(\Gamma_0)$ one.

A similar procedure based on the method of fundamental solutions (MFS) has been presented by T. Wei and Y.G. Chen in [28]; the main difference is that their approach assumes that the exact solution is close enough to a finite dimensional subspace of $H = \{u \in H^p(\Omega) : \Delta u \equiv 0\} (p > 3/2)$, which actually imposes a high smoothness *a priori* condition over the exact solution.

The paper is organized as follows. Section 2 is devoted to formulating the inverse problem in the framework of this methodology. In section 3, the regularization strategy is formally presented, and optimal accuracy is proven. The link with the factorization method in [16] is shown in section 3 as well, suggesting discretization methods for the implementation of the AD solution in complex geometries of Ω. Finally, in section 4, the methodology is applied when Ω is a cylinder, and a numerical example is given, comparing with the Tikhonov regularization strategy, the most common method employed to solve linear ill-posed problems, which behaves well in less ill-posed problems, but requires strong regularity on the solution to be optimal in severely ill-posed cases.

2. Operational Formulation of boundary data completion, and admissible data definition

Present section is devoted to formally set the operational equation that defines the inverse problem. Through this section and the next one, $\Omega \subset \mathbb{R}^{n+1}$ $(n \geq 1)$ will be a domain with a piecewise smooth boundary $\partial\Omega$, Γ_0 and Γ_1 will denote two disjoint C^∞ surfaces of dimension n in $\partial\Omega$, and $\Sigma = \partial\Omega \setminus (\Gamma_0 \cup \Gamma_1)$. It will also be assumed that the euclidean distance between closures of Γ_0 and Γ_1 is not null, and that the interior of Σ in the $\partial\Omega$ is regular too. Let us define $D^\alpha f$ as the Sobolev partial derivative of order $\alpha = (\alpha_1, ..., \alpha_n)$ of $f \in L^1_{loc}(\Omega)$, $\mathrm{H}^s(\Omega)$ the Sobolev space $W^{s,2}$, and $H^1_0(\Omega)$ the closure in the H^1-norm of the test function $\mathcal{D}(\Omega)$ (functions infinitely differentiable with compact support in Ω). The L^2-norm, and the standard operator norm will be denoted by the symbol $\|\cdot\|$. When X is a normed space, the open ball in X centered at x with radius r will be denoted by $\mathcal{B}_X(x,r)$; notation $\mathrm{cl}_X(M)$ is reserved to the closure of the set M in the topology of X. As particular cases, notation conventions $\mathcal{B}_X(0,r) = \mathcal{B}_X(r)$, and $\overline{\mathcal{B}}_X(x,r) = \mathrm{cl}_X(\mathcal{B}_X(x,r))$ are established.

Following subspaces of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{\frac{1}{2}}(S)$ are set forth:

$$E_0 = \left\{v \in \mathrm{H}^1(\Omega) : v|_\Sigma = 0\right\}, \; E_{00} = \left\{v \in \mathrm{H}^1(\Omega) : v|_{\Sigma\cup\Gamma_1} = 0\right\},$$

and

$$E^{\frac{1}{2}}(\Gamma_i) = \left\{\phi \in \mathrm{H}^{\frac{1}{2}}(\Gamma_i) : \exists v \in E_0 : v|_{\Gamma_i} = \phi\right\}, \quad i = 0 \text{ or } 1.$$

Notation $E^{-\frac{1}{2}}(\Gamma_i)$ will be reserved for the dual of $E^{\frac{1}{2}}(\Gamma_i)$. By Poincaré inequality [19], spaces $E_0(\Omega)$, and $E_{00}(\Omega)$ will be provided with the equivalent norm induced by the inner product:

$$(\nabla u, \nabla v)_{\mathrm{L}^2} = \int_\Omega \nabla u \nabla v dx = \int_\Omega \sum_{k=1}^{n+1} \frac{du}{dx_k}\frac{dv}{dx_k} dx.$$

Now, consider boundary value problem (6)-(9), and definition 2.1, slightly differing from the usual definition of *weak solution* :

$$\Delta u \equiv 0 \quad in\ \Omega, \tag{6}$$
$$u|_\Sigma = 0, \tag{7}$$
$$u|_{\Gamma_1} = \Phi, \tag{8}$$
$$\left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_0} = \psi. \tag{9}$$

Definition 2.1. For a given couple of boundary conditions (Φ, ψ) in $E^{\frac{1}{2}}(\Gamma_1) \times \mathrm{L}^2(\Gamma_0)$, function u belonging to E_0 is a *weak solution* of (6)-(9) if equation (10) and boundary condition $u|_{\Gamma_1} = \Phi$ are both fulfilled.

$$\int_\Omega \nabla u \nabla v dx = \int_{\Gamma_0} \psi v dS, \quad \forall v \in E_{00}. \tag{10}$$

The set of the standard test function $\mathcal{D}(\Omega)$ is contained in E_{00}, so that the weak solution is also a solution in the sense of [27]. The existence, uniqueness, and continuous dependence on the boundary data of the weak solution to problem (6)-(9) are proven in an analogous way as Mijailov did in [27] for elliptic boundary value problems.

Definition 2.2 (Admissible Cauchy data)**.** Data (ϕ, ψ) in $\mathrm{H}^{\frac{1}{2}}(\Gamma_0) \times L^2(\Gamma_0)$ will be called an *admissible data* for the Cauchy problem (1)-(4), if there exist Φ in $E^{\frac{1}{2}}(\Gamma_1)$, and u in $\mathrm{H}^1(\Omega)$ such that: u solves eq.(6)-(9) in the sense of Definition 2.1 for the pair (Φ, ψ), and the Dirichlet condition $u|_{\Gamma_0} = \phi$ is also fulfilled .

Function Φ will be called a *solution* of the Dirichlet boundary data completion problem for (1)-(4).

In order to establish the direct problem that defines the admissible Cauchy data class, functions u_1 and u_2 will denote solutions of problem (6)-(9) for boundary data $(\Phi, \psi \equiv 0)$ and $(\Phi \equiv 0, \psi)$, respectively. Function u_1 depends only on Φ, u_2 on ψ, and the solution of the auxiliary boundary problem (6)-(9) is given by $u = u_1 + u_2$; hence, identifying the Dirichlet condition in (1)-(4) can be approached as a first kind linear inverse problem via the Dirichlet to Dirichlet operator A, from Γ_1 to Γ_0, and the Neumann to Dirichlet operator B defined in Γ_0:

$$\begin{array}{ccc} A : E^{\frac{1}{2}}(\Gamma_1) & \longrightarrow & \mathrm{L}^2(\Gamma_0) \\ \Phi & \mapsto & u_1|_{\Gamma_0} \end{array}, \qquad \begin{array}{ccc} B : L^2(\Gamma_0) & \longrightarrow & \mathrm{L}^2(\Gamma_0) \\ \psi & \mapsto & u_2|_{\Gamma_0} \end{array}.$$

Each function u_1 depending on Φ, and u_2 depending on ψ solves a well-posed boundary data problem, and the trace operator is compact from $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^2(\Gamma_0)$ [18, 19, 27]; hence the linear operators A and B are compact. We extend the range of both operators A and B to $\mathrm{L}^2(\Gamma_0)$ instead of $E^{\frac{1}{2}}(\Gamma_0)$ because, taking into account the noise in the measurements, we consider L^2-norms in the regularization stage.

It is a well known fact that Cauchy problems have a unique solution [5, 26]. Therefore, $A\Phi = 0$ implies that u_1 solves the Cauchy problem for a null Cauchy data given on Γ_0, and hence A is injective. On the other hand, let us prove that B is injective, self-adjoint and positive defined: choose v in (10) as the solution of problem (6)-(9) for a third boundary data $(\Phi \equiv 0, \overline{\psi})$; B is self-adjoint follows immediately from Definition 2.1; taking $\overline{\psi} = \psi$ proves that B is positive defined, and under same equality the assumption $B\phi = 0$ implies that solution of (6)-(9) is null for $(\Phi \equiv 0, \psi)$, which proves that B is injective.

Now, by definition of A and B, the set $\mathbb{M}$ of all admissible data for the Cauchy problem can be defined equivalently as the linear space:

$$\mathbb{M} = \left\{ (A\Phi + B\psi, \psi) : \Phi \in E^{\frac{1}{2}}(\Gamma_1),\ \psi \in \mathrm{L}^2(\Gamma_0) \right\}. \tag{11}$$

The forward problem from which boundary data completion is established is to determine the trace over Γ_0 of the solution of boundary problem (6)-(9) from given mixed boundary data (Φ, ψ), which in the form of an abstract equation means to evaluate $A\Phi + B\psi$. Then we set the inverse problem of interest as follows:

$$A\Phi = \rho, \quad \rho = \phi - B\psi. \tag{12}$$

Characterizing $E^{\frac{1}{2}}(\Gamma_1)$ represents an essential step toward an admissible data characterization. The norm in $E^{\frac{1}{2}}(\Gamma_1)$ is defined by $\|\Phi\|_{E^{\frac{1}{2}}} = \|\nabla u_1\|$. If P is the Dirichlet to Neumann operator in Γ_1 for the boundary data problem (6)-(9) with $\psi \equiv 0$: $P : \Phi \mapsto \frac{\partial u_1}{\partial \nu}\big|_{\Gamma_1}$, then, in virtue of Green's formula:

$$\|\Phi\|^2_{E^{\frac{1}{2}}} = \langle \Phi, P\Phi \rangle_{E^{\frac{1}{2}}(\Gamma_1) \times E^{-\frac{1}{2}}(\Gamma_1)}.$$

All the properties of P are studied in detail in [16].

3. Regularization Strategy

In later sections it is shown that the Dirichlet to Dirichlet mapping A can be extended in the L^2 sense for cylindrical domains and be considered as a bounded mapping from $\mathrm{L}^2(\Gamma_1)$ to $\mathrm{L}^2(\Gamma_0)$. However, it could happen that A coud not be extended in the general case. The possibility to extend A is helpful for the regularization stage, so that, it will be considered as an additional *a priori* information when possible. Using a shorter notation, $\mathrm{D}(A)$ will denote the domain of A, where $\mathrm{D}(A)$ will be understood as $\mathrm{L}^2(\Gamma_1)$ or $E^{\frac{1}{2}}(\Gamma_1)$, depending on whether A can be extended or not: $A : \mathrm{D}(A) \to \mathrm{L}^2(\Gamma_0)$, meaning that the symbol A will be employed to denote the original operator defined in previous section, or its extension to $\mathrm{L}^2(\Gamma_0)$ as appropriate.

Now, let u_1, and u_2 be as before. By replacing u by u_1, and v by u_2 in (10) it immediately follows that $\int_\Omega \nabla u_1 \nabla u_2 dx = 0$, and replacing both u and v by u_2 gives that $\|\nabla u_2\|^2 = \int_{\Gamma_0} \psi B \psi dS$. Hence, analogously to how norm $\|\cdot\|_{E^{\frac{1}{2}}(\Gamma_1)}$ was defined, the natural norm over the admissible data set $\mathbb{M}$ in eq (11) is given by the H^1-norm of the corresponding weak solution of the BVP (1)-(4): $\|\nabla(u_1+u_2)\|^2 = \|\Phi\|^2_{E^{\frac{1}{2}}} + \int_{\Gamma_0} \psi B \psi dS$, with $\Phi = A^{-1}(\phi - B\psi)$. However, B is bounded; then, the really natural norm over the admissible data class in this case is a stronger one defined by:

$$\|(\phi,\psi)\|_{\mathrm{ad}} = \sqrt{\|\Phi\|^2_{E^{\frac{1}{2}}(\Gamma_1)} + \|\psi\|^2}, \quad \Phi = A^{-1}(\phi - B\psi).$$

For the inverse problem (12), a measured (or perturbed in synthetic examples) Cauchy data $(\tilde{\phi}, \tilde{\psi})$ will be known instead of the exact one $(\phi^\dagger, \psi^\dagger)$, with a L^2-noise level of order $\delta > 0$, meaning that the inverse problem to regularize is

$$A\Phi^\dagger = \phi^\dagger - B\psi^\dagger; \quad \left\|(\phi^\dagger, \psi^\dagger)\right\|_{\mathrm{ad}} < \infty, \quad \left\|\phi^\dagger - \tilde{\phi}\right\| \le \delta, \quad \left\|\psi^\dagger - \tilde{\psi}\right\| \le \delta. \tag{13}$$

Then, defining $\mathbb{M}^\delta = \{(\phi,\psi) \in \mathbb{M} : \left\|(\phi - \tilde{\phi}, \psi - \tilde{\psi})\right\| \le \sqrt{2}\delta\}$, we have that (13) implies

$$(\phi^\dagger, \psi^\dagger) \in \mathbb{M}^\delta \cap \overline{\mathcal{B}}_{\mathrm{ad}}(K) \;\text{ for a given } K > 0. \tag{14}$$

Due to the compact injection of $E^{\frac{1}{2}}(\Gamma_1)$ in $\mathrm{L}^2(\Gamma_1)$, a quasi solution regularization scheme is immediately suggested from (14) to the inverse problem (12) when $\mathrm{D}(A) = \mathrm{L}^2(\Gamma_1)$. As it will be shown shortly, the admissible data solution (**AD** solution) is indeed a quasi solution scheme with a slightly more regular definition of (14) as cornerstone. For a given K as *a priori* information, a natural candidate to be a regularization of $\Phi^\dagger$, whether the distance from measurement to $\mathrm{cl}_{\mathrm{ad}}(\mathcal{B}(K))$ can be reached at some element $(\phi^\delta, \psi^\delta)$, is of course $\Phi^\delta = A^{-1}(\phi^\delta - B\psi^\delta)$. What AD methodology is looking for in our case, are sufficient conditions to guarantee the existence of $(\phi^\delta, \psi^\delta)$, and simultaneously that Φ^δ tends to $\Phi^\dagger$ when δ tends to 0.

The AD solution of problem (12) requires the L^2 minimum distance projection from perturbed Cauchy data over the admissible data set, because of the noise level in (13) is given in the L^2-norm as well. Unfortunately, the problem to find the aforementioned projection is ill-posed, because generally the admissible data set is not a closed space in $\mathrm{L}^2(\Gamma_0) \times \mathrm{L}^2(\Gamma_0)$. To get a good enough behaviour of the projection some *a priori* information is required. In our case, convexity of the compact subset to which the solution belongs is that required information.

3.1. **Setting the strategy**

As *a priori* information it will be required that the exact couple $(\phi^\dagger, \psi^\dagger)$ belongs to a convex subset M of admissible data of the form

$$M = \{(A\Phi + B\psi, \psi) \in \mathbb{M} : \Phi \in Z_1, \quad \psi \in Z_0\}; \tag{15}$$

where Z_0 is a convex subset of $\mathrm{L}^2(\Gamma_0)$, and Z_1 is compact and convex in $\mathrm{D}(A)$ ($\mathrm{L}^2(\Gamma_1)$ or $E^{\frac{1}{2}}(\Gamma_1)$), the required *a priori* for the construction of a quasi solution of the inverse problem $A\Phi = \rho$.

The minimum L^2-distance projection from $(\tilde{\phi}, \tilde{\psi})$ to $\mathrm{cl}_{(\mathrm{L}^2(\Gamma_0))^2}(M)$ exists, and is unique since $\mathrm{cl}_{(\mathrm{L}^2(\Gamma_0))^2}(M)$ is closed and convex [20, sec. 3.12]. Let $(\phi^\delta, \psi^\delta)$ be the aforementioned projection, also defined by

$$(\phi^\delta, \psi^\delta) = \underset{(\phi,\psi) \in \mathrm{cl}_{(\mathrm{L}^2(\Gamma_0))^2}(M)}{\arg\min} \left(\left\|\phi - \tilde{\phi}\right\|^2 + \left\|\psi - \tilde{\psi}\right\|^2 \right). \tag{16}$$

We shall see that $(\phi^\delta, \psi^\delta)$ is in fact an admissible data, and that the corresponding Dirichlet data $\Phi^\delta = A^{-1}(\phi^\delta - B\psi^\delta)$ converges to the exact solution $\Phi^\dagger$ when δ tends to 0.

Lemma 3.1. *The projection* $(\phi^\delta, \psi^\delta)$ *verifies* $\left\|(\phi^\dagger - \phi^\delta, \psi^\dagger - \psi^\delta)\right\| \le \delta_1$*, where*

$$0 \le \delta_1 = \sqrt{2\delta^2 - \left\|\phi^\delta - \tilde{\phi}\right\|^2 - \left\|\psi^\delta - \tilde{\psi}\right\|^2} \le \sqrt{2}\delta.$$

Proof. First,

$$\begin{aligned} \|\phi^\dagger - \phi^\delta\|^2 + \|\psi^\dagger - \psi^\delta\|^2 = {} & \left\|\phi^\dagger - \tilde{\phi}\right\|^2 + \left\|\psi^\dagger - \tilde{\psi}\right\|^2 - \left\|\phi^\delta - \tilde{\phi}\right\|^2 - \left\|\psi^\delta - \tilde{\psi}\right\|^2 \\ & +2\left(\left\langle \phi^\dagger - \phi^\delta, \tilde{\phi} - \phi^\delta\right\rangle + \left\langle \psi^\dagger - \psi^\delta, \tilde{\psi} - \psi^\delta\right\rangle\right), \end{aligned} \tag{17}$$

but, $\mathrm{cl}_{(\mathrm{L}^2(\Gamma_0))^2}(M)$ is convex and $(\phi^\delta, \psi^\delta)$ must ensure (see [20, sec. 3.12])

$$\left\langle \phi - \phi^\delta, \tilde{\phi} - \phi^\delta\right\rangle + \left\langle \psi - \psi^\delta, \tilde{\psi} - \psi^\delta\right\rangle \le 0, \quad \forall \ (\phi, \psi) \in \mathrm{cl}_{(\mathrm{L}^2(\Gamma_0))^2}(M). \tag{18}$$

The result follows from (17), (18), and because $(\phi^\dagger, \psi^\dagger) \in \mathrm{cl}_{(\mathrm{L}^2(\Gamma_0))^2}(M)$. □

In other words, projection $(\phi^\delta, \psi^\delta)$ can be thought as a perturbation of $(\phi^\dagger, \psi^\dagger)$ with smaller or equal noise error than $(\tilde{\phi}, \tilde{\psi})$. The immediate step is to solve (19) instead of (13), with δ_1 as in Lemma 3.1:

$$A\Phi^\dagger = \phi^\dagger - B\psi^\dagger; \quad \left\|(\phi^\dagger - \phi^\delta, \psi^\dagger - \psi^\delta)\right\| \le \delta_1. \tag{19}$$

Lemma 3.2. *The projection* $(\phi^\delta, \psi^\delta)$ *in Lemma 3.1 is an admissible data, and* $\Phi^\delta = A^{-1}(\phi^\delta - B\psi^\delta)$ *converges to* $\Phi^\dagger$ *when* δ *tends to 0.*

Proof. Let $\{(\phi_n, \psi_n)\}_{n\in\mathbb{N}}$ be a sequence in M that converges to $(\phi^\delta, \psi^\delta)$. Denoting $\rho_n = \phi_n - B\psi_n$ ($\rho^\delta = \phi^\delta - B\psi^\delta$), and $\Phi_n = A^{-1}\rho_n \in Z_1$, it immediately follows that we can choose $\Phi^\delta = A^{-1}\rho^\delta$ in Z_1 as any accumulation point of $\{\Phi_n\}_{n\in\mathbb{N}}$, which exists since Z_1 is compact in $\mathrm{D}(A)$. By continuity of A and B we have $A\Phi^\delta + B\psi^\delta = \phi^\delta$. So $(\phi^\delta, \psi^\delta)$ is admissible and uniqueness of Φ^δ holds true in virtue of the injectivity of A.

To finish the proof, for $\delta > 0$ let N_δ be the smallest natural such that

$$\forall n \ge N_\delta : \quad \left\|\Phi_{N_\delta} - \Phi^\delta\right\|_{\mathrm{D}(A)} \le \delta, \text{ and } \quad \left\|\phi_{N_\delta} - \tilde{\phi}\right\| + \left\|\psi_{N_\delta} - \tilde{\psi}\right\| \le \frac{\delta}{1 + \|B\|},$$

then, by triangular inequality

$$\left\|\Phi^\delta - \Phi^\dagger\right\|_{\mathrm{D}(A)} \le \delta + \left\|\Phi_{N_\delta} - \Phi^\dagger\right\|_{\mathrm{D}(A)}.$$

But, Φ_{N_δ} belongs to the intersection of Z_1, and the following set

$$Z^\delta_{(\tilde{\phi},\tilde{\psi})} = \{\Phi \in \mathrm{D}(A) : \|A\Phi - \tilde{\rho}\| \le \delta\}, \quad \tilde{\rho} = \tilde{\phi} - B\tilde{\psi}.$$

Hence, Φ_{N_δ} converges to $\Phi^\dagger$ in virtue of Theorem 1 in [4, ch.6, sec 1]. □

Lemmas 3.1-3.2 actually show the heart of the methodology. We are taking full advantage of the available *a priori* information to regularize the minimum distance projection from measured data to the admissible data with a quasi solution regularization scheme.

There is more than one way to select Z_1. Mainly, we are going to work with the one which is related to the worst case error definition in [1, sec1.3]. Let X and Y denote Banach spaces, X_1 a dense subspace of X endowed with the norm $\|\cdot\|_1$, stronger than the norm in X, $T : X \to Y$ a bounded linear operator and $\omega_T(\delta, K, \|\cdot\|_1)$ the worst case error for T corresponding to the noise level $\delta > 0$ and *a priori* information $\left\|x^\dagger\right\|_1 \le K$ in the framework of the inverse problem $Tx^\dagger = y^\dagger$:

$$\omega_T(\delta, K, \|\cdot\|_1) = \sup\{\|x\|_X : x \in X_1, \ \ \|Tx\|_Y \le \delta, \ \ \|x\|_1 \le K\}.$$

In our case, $\mathrm{D}(A)$ will play the role of X in the previous definition of the worst case error, $Y = \mathrm{L}^2(\Gamma_0)$, and X_1 will be replaced by F, a subspace with compact injection over $\mathrm{D}(A)$; for congruence in the notation $\|\cdot\|_F$ replaces $\|\cdot\|_1$. When A can be extended and $\mathrm{D}(A) = \mathrm{L}^2(\Gamma_1)$ the immediate choice of F is $E^{\frac{1}{2}}(\Gamma_1)$. Otherwise, whether $\mathrm{D}(A) = E^{\frac{1}{2}}(\Gamma_1)$, we can choose F by considering extra *a priori* information as in section 3.2, or, for example when is also assumed that the exact solution $\Phi^\dagger$ belongs to a well known finite dimensional subspace of $E^{\frac{1}{2}}(\Gamma_1)$.

In the same way as norm $\|(\cdot,\cdot)\|_{\mathrm{ad}}$, we define over the subspace of admissible data

$$\mathbb{M}_F = \left\{(A\Phi + B\psi, \psi) : \Phi \in F, \psi \in L^2(\Gamma_0)\right\}$$

its own norm, stronger than $\|(\cdot,\cdot)\|_{\mathrm{ad}}$:

$$\|(\phi,\psi)\|_0 = \sqrt{\|\Phi\|_F^2 + \|\psi\|^2}; \quad \Phi = A^{-1}(\phi - B\psi).$$

In previous framework, $Z_1 = \overline{\mathcal{B}}_F(K)$, and Z_0 such that $M = \overline{\mathcal{B}}_{\mathrm{ad}}(K)$ i.e

First assumption of *a priori* information (1^{st} assumption):
An upper bound K of $\left\|(\phi^\dagger,\psi^\dagger)\right\|_0$ will be given as *a priori* information.

Now, by how it has been built, the AD regularized solution Φ^δ must be optimal in the sense of Definition 1.18 in [1] for the just above assumed *a priori* information over the exact Cauchy data.

Theorem 3.3. *Under the 1^{st} assumption and if $(\tilde{\phi},\tilde{\psi})$ does not belong to $\mathrm{cl}_{(\mathrm{L}^2(\Gamma_0))^2}(M)$ then, function Φ^δ defined in Lemma 3.2 is an optimal regularized solution of the linear inverse problem of the first kind* (19) *in the sense of Definition 1.18 in [1], for the error $\sqrt{2}\delta\max\{1,\|B\|\}$ in the data, and a priori information $\left\|\Phi^\dagger\right\|_F \le 2K$.*

Proof. By Lemmas 3.1 and 3.2 it happens that, for any $n \ge N_{\delta_1}$, $\Phi^\dagger - \Phi_n \in \overline{\mathcal{B}}_F(K)$, and

$$\left\|A(\Phi^\dagger - \Phi_n)\right\| \quad \le \quad \delta_1 \max\{1,\|B\|\}\left(1 + \left\|\phi^\delta - \phi_n\right\| + \left\|\psi^\delta - \psi_n\right\|\right); \tag{20}$$

then, defining $\epsilon_{1,n} = \left\|\phi^\delta - \phi_n\right\| + \left\|\psi^\delta - \psi_n\right\| \to 0$, and $\epsilon_{2,n} = \left\|\Phi_n - \Phi^\delta\right\|_{\mathrm{D}(A)} \to 0$:

$$\left\|\Phi^\dagger - \Phi^\delta\right\|_{\mathrm{D}(A)} \le \left\|\Phi^\dagger - \Phi_n\right\|_{\mathrm{D}(A)} + \epsilon_{2,n} \le \omega_A((1+\epsilon_{1,n})\delta_1\max\{1,\|B\|\}, K, \|\cdot\|_F) + \epsilon_{2,n}. \tag{21}$$

However, $(1+\epsilon_{1,n})\delta_1 < \sqrt{2}\delta$ for any large enough n since $(\tilde{\phi},\tilde{\psi}) \notin \mathrm{cl}_{(\mathrm{L}^2(\Gamma_0))^2}(M)$. □

Remark 3.4. Best case for the 1^{st} assumption is $K = \left\|(\phi^\dagger,\psi^\dagger)\right\|_0$.

Even if Theorem 3.3 in the present form is not helpful in numerical calculations, it shows that the regularization error under this regularization methodology is optimal under the 1^{st} assumption. It is well known that inverse problem (19) is exponentially ill-posed, and as it shall be seen in section 4, last definition of Z_1 is equivalent to say that $\Phi^\dagger$ satisfies a *logarithmic source condition* ($\Phi^\dagger = -\ln^{-p}(A^\star A)\zeta$, $\|\zeta\| \le K_1$, $p > 0$, where $A^\star$ denotes the A adjoint). The result that $\omega_A(\delta,K,F)$ must be of order $O(-\ln^{-p}(\delta))$ is also shown in [24,25], and other recent studies on convergence rate estimates can be found in [3,21].

3.2. Additional *a priori* information, and link with the Tikhonov regularization

First practical issue of the AD solution is that there is not an equivalent Lagrangian formulation of the problem, because M is not closed in the L^2-norm. In fact, the imposed constraint by the 1^{st} assumption may be not defined in $(\phi^\delta,\psi^\delta)$ i.e the projection $(\phi^\delta,\psi^\delta)$ could be a non-regular point for the constraint defining M in the sense of [20]. The present section is devoted to provide conditions for which the equivalent Lagrangian formulation of the main optimization problem exists under the 1^{st} assumption. The main motivation to do this is to get an approach that allows to turn the value of K from *a priori* information into a regularization parameter when K is unknown, preserving the optimality of the solution for some regularization parameter, as is discussed in [11, 21].

Stronger *a priori* information over solution $\Phi^\dagger$ will be required through this section in order to get an equivalent optimization problem to (16), where the projection $(\phi^\delta, \psi^\delta)$ is a regular point for the corresponding constraint.

Consider the Neumann to Dirichlet operator defined in Γ_1 by $Q : \Psi \in \mathrm{L}^2(\Gamma_1) \mapsto u_3|_{\Gamma_1}$, where u_3 is the solution of problem (22)-(25) in an analogous sense to Definition 2.1:

$$\begin{aligned} \Delta u_3 &\equiv 0 \quad in\ \Omega, && (22)\\ u_3|_\Sigma &= 0, && (23)\\ \left.\frac{\partial u_3}{\partial \nu}\right|_{\Gamma_1} &= \Psi, && (24)\\ \left.\frac{\partial u_3}{\partial \nu}\right|_{\Gamma_0} &= 0. && (25) \end{aligned}$$

Just like in the case of operator B, Q is a compact mapping from $\mathrm{L}^2(\Gamma_1)$ to itself, and bounded from $\mathrm{L}^2(\Gamma_1)$ to $E^{\frac{1}{2}}(\Gamma_1)$. As an operator in $\mathrm{L}^2(\Gamma_1)$ to itself it is also self adjoint and positive defined. Moreover, it is clear that P is the left inverse of Q ($I = PQ$), and $QP\Phi = \Phi$ if $P\Phi$ is a regular distribution ($P\Phi \in \mathrm{L}^2(\Gamma_1)$).

Remark 3.5. Uniqueness of solution u_3 is true in this case by the boundary condition (23), without which the solution is unique up to an additive constant. This fact makes necessary to be careful if the methodology needs to be extended to cases where $\partial\Omega$ is composed only by the two regular and smooth surfaces Γ_0, and Γ_1. The ECG inverse problem, for instance, is so.

Define $E_s^{\frac{1}{2}}(\Gamma_1) = E^{\frac{1}{2}}(\Gamma_1) \cap Q^s(\mathrm{L}^2(\Gamma_1))$ $(s > 0)$ as follows

$$E_s^{\frac{1}{2}}(\Gamma_1) = \left\{\Phi \in E^{\frac{1}{2}}(\Gamma_1) : \Phi = Q^s\zeta, \quad \zeta \in \mathrm{L}^2(\Gamma_1)\right\}, \quad s > 0; \quad \|\Phi\|_{E_s^{\frac{1}{2}}(\Gamma_1)} = \|\zeta\|. \tag{26}$$

Injection $E_{s_2}^{\frac{1}{2}}(\Gamma_1) \to E_{s_1}^{\frac{1}{2}}(\Gamma_1)$ is compact as soon as $s_2 > s_1 > \frac{1}{2}$.

After the definition of $E_s^{\frac{1}{2}}(\Gamma_1)$ $(s > 0)$ we are ready to work with a stronger condition than the 1^{st} assumption. Now we will consider the following definition of the space F:

$$F = E_s^{\frac{1}{2}}(\Gamma_1) \ \text{ where } \ s \geq \frac{1}{2} \ \text{ if } \ \mathrm{D}(A) = \mathrm{L}^2(\Gamma_1), \ \text{ and } \ s > \frac{1}{2} \ \text{ if } \ \mathrm{D}(A) = E^{\frac{1}{2}}(\Gamma_1).$$

The 1^{st} assumption is rewritten as

Second assumption of *a priori* information (2^{nd} assumption)

There exists $\zeta^\dagger \in \mathrm{L}^2(\Gamma_1)$ such that $\Phi^\dagger = Q^s\zeta^\dagger$ (s depending on $\mathrm{D}(A)$), and an *a priori* given $K > 0$ such that

$$\|(\phi, \psi)\|_0 = \sqrt{\|\zeta^\dagger\|^2 + \|\psi^\dagger\|^2} \leq K.$$

Through the 2^{nd} assumption we have that solving problem replace (16) is equivalent to solve

$$(\zeta^\delta,\psi^\delta) = \begin{array}{cc} \underset{\zeta\in L^2(\Gamma_1),\ \psi\in L^2(\Gamma_0)}{\arg\min} & \left\|AQ^s\zeta + B\psi - \tilde{\phi}\right\|^2 + \left\|\psi - \tilde{\psi}\right\|^2 \\ \text{such that} & \|\zeta\|^2 + \|\psi\|^2 - K^2 \le 0 \end{array} \tag{27}$$

and define the AD solution by

$$\Phi^\delta = Q^s\zeta^\delta.$$

Now, solution of (27) is a regular point for its constraint because $(\zeta^\delta,\psi^\delta)$ satisfies this constraint. The objective functional and the constraint are both convex and have Gateaux derivatives linear in their increments; then, by the generalized Kuhn-Tucker theorem in the classic optimization theory [20], there exists $\alpha_K \ge 0$ (a Lagrange multiplier) such that solving problem (27) is equivalent to solving its Lagrangian formulation for $\alpha = \alpha_k$:

$$(\zeta^{\delta,\alpha},\psi^{\delta,\alpha}) = \underset{(\zeta,\psi)\in L^2(\Gamma_1)\times L^2(\Gamma_0)}{\arg\min} \quad J_\alpha(\zeta,\psi) \tag{28}$$

where $J_\alpha(\zeta,\psi) = \left\|AQ^s\zeta + B\psi - \tilde{\phi}\right\|^2 + \left\|\psi - \tilde{\psi}\right\|^2 + \alpha\left(\|\zeta\|^2 + \|\psi\|^2\right)$.

Define the operator W by $W : \psi \mapsto -\left.\frac{\partial u_2}{\partial\nu}\right|_{\Gamma_1}$ (u_2, and u_1 the same solutions of (6)-(9) that define B, and A respectively). Solution u_2 continuously depends on ψ in $\mathrm{L}^2(\Gamma_0)$, and the outward normal derivative operator over the boundary of Γ is continuous from $\mathrm{H}^1(\Omega)$ to $E^{-\frac{1}{2}}(\Gamma_1)$, meaning that W is bounded from $\mathrm{L}^2(\Gamma_0)$ to $E^{-\frac{1}{2}}(\Gamma_1)$. Moreover, with almost no change on the existence, uniqueness, and continuity demonstrations, solution u_3 defining Q can be extended to Ψ in $E^{-\frac{1}{2}}(\Gamma_1)$. This means that operator Q can be extended to the compact operator $\bar{Q} : E^{-\frac{1}{2}}(\Gamma_1) \to \mathrm{L}^2(\Gamma_1)$; then, operator $\bar{Q}^sW$ $(s>0)$ is compact from $\mathrm{L}^2(\Gamma_0)$ to $\mathrm{L}^2(\Gamma_1)$. Now, when $\Phi = Q^s\zeta$ with ζ in $\mathrm{L}^2(\Gamma_1)$, again by Green's formula and because u_1 is orthogonal to u_2 with the inner product $(\nabla\cdot,\nabla\cdot)_{\mathrm{L}^2}$, it happens that:

$$\int_{\Gamma_0}(AQ^s\zeta)\,\psi dS = \langle Q^s\zeta, W\psi\rangle_{E^{\frac{1}{2}}(\Gamma_1)\times E^{-\frac{1}{2}}(\Gamma_1)} = \left\langle \zeta, \bar{Q}^sW\psi\right\rangle_{E^{\frac{1}{2}}(\Gamma_1)\times E^{-\frac{1}{2}}(\Gamma_1)} = \int_{\Gamma_1}\zeta\bar{Q}^sW\psi dS, \quad \forall\psi\in\mathrm{L}^2(\Gamma_0); \tag{29}$$

So clearly, $\bar{Q}^sW$ is the L^2 adjoint of $A_s = AQ^s$ $(A_s^\star = \bar{Q}^sW)$. This way, the J_α Gateaux derivative in (ζ,ψ), and applied to $\mathbf{x} = (\mathbf{x}_1,\mathbf{x}_2)$ is given by

$$\begin{array}{rcl} DJ_\alpha(\zeta,\psi)(\mathbf{x}) & = & \int_{\Gamma_0}\left[A_s^\star\left(A_s\zeta + B\psi - \tilde{\phi}\right) + \alpha\zeta\right]\mathbf{x}_1 dS \\ & & + \int_{\Gamma_0}\left[B\left(A_s\zeta + B\psi - \tilde{\phi}\right) + (1+\alpha)\psi - \tilde{\psi}\right]\mathbf{x}_2 dS, \end{array}$$

where I is the identity in $\mathrm{L}^2(\Gamma_0)$. Hence, $(\zeta^{\delta,\alpha},\psi^{\delta,\alpha})$ solves:

$$(A_s^\star A_s + \alpha I)\,\zeta + A_s^\star B\psi = A_s^\star\tilde{\phi}, \tag{30}$$

$$BA_s\zeta + \left(B^2 + (1+\alpha)I\right)\psi = B\tilde{\phi} + \tilde{\psi}. \tag{31}$$

Parameter α is a Lagrange multiplier, meaning that the α value making equivalent the optimization problems is null $(\alpha_K = 0)$ if there is $(\tilde{\zeta},\tilde{\psi})$ in $\overline{\mathcal{B}}_{\mathrm{L}^2(\Gamma_1)\times\mathrm{L}^2(\Gamma_0)}(K)$ such that $\tilde{\phi} = A_s\tilde{\zeta} + B\tilde{\psi}$. Otherwise α_K must fulfill

$$\left\|\zeta^{\delta,\alpha_K}\right\|^2 + \left\|\psi^{\delta,\alpha_K}\right\|^2 = K^2, \quad \left\|\Phi^{\delta,\alpha_K}\right\|_{E_s^{\frac{1}{2}}(\Gamma_1)} = \left\|\zeta^{\delta,\alpha_K}\right\|, \quad \Phi^\delta = \Phi^{\delta,\alpha_K}. \tag{32}$$

Normal equations (30)-(31) provide the first corollary to Theorem 3.3.

Corollary 3.6. *If the 2^{nd} assumption is fulfilled, then, for any $\alpha > 0$ the pair $(\Phi^{\delta,\alpha}, \psi^{\delta,\alpha})$ is defined by* (33)-(34)

$$\psi^{\delta,\alpha} = \left\{B\left[I - A_s L_{s,\alpha}^{-1} A_s^\star\right] B + (1+\alpha)I\right\}^{-1} \left\{B\left[I - A_s L_{s,\alpha}^{-1} A_s^\star\right]\tilde{\phi} + \tilde{\psi}\right\}, \tag{33}$$

$$\Phi^{\delta,\alpha} = Q^s L_{s,\alpha}^{-1} A_s^\star \left(\tilde{\phi} - B\psi^{\delta,\alpha}\right) \tag{34}$$

where $L_{s,\alpha} = A_s^\star A_s + \alpha I$. Moreover, under the 2^{nd} assumption and if $(\tilde{\phi}, \tilde{\psi})$ does not belong to $\mathrm{cl}_{(\mathrm{L}^2(\Gamma_0))^2}(M)$, there exists $\alpha_K > 0$ such that

$$\left\|\Phi^\dagger - \Phi^{\delta,\alpha_K}\right\| \le \omega_A\left(\sqrt{2}\delta\max\{1, \|B\|\}, 2K, E_s^{\frac{1}{2}}(\Gamma_1)\right).$$

Proof. To get (33)-(34) it is only needed to algebraically solve system (30)-(31). A well known fact is that $L_{s,\alpha}$ is invertible. By a singular value decomposition it is also easy to see that $I - A_s L_{s,\alpha}^{-1} A_s^\star$ is positive defined, implying that any positive real belongs to the resolvent of $B\left(I - A_s L_{s,\alpha}^{-1} A_s^\star\right)B$, which finally proves that $\Phi^{\delta,\alpha}$ and $\psi^{\delta,\alpha}$ are well defined by (33)-(34). □

As usual in Lagrangian formulations for inverse problems, eq (32) says that $K = \left\|(\Phi^\dagger, \psi^\dagger)\right\|_0$ is what really matters to know *a priori* in this kind of approach. One has to look for the smallest value of K that makes sense, and minimize the worst case error $\omega_A\left(\sqrt{2}\delta\max\{1, \|B\|\}, 2K, E^{\frac{1}{2}}\right)$ for a given $\delta > 0$. The dependency of α on K actually makes that parameter α can be considered as *a priori* information when K is known; however, via the Lagrangian formulation, when K is unknown and $\left\|(\Phi^\dagger, \psi^\dagger)\right\|_0 < \infty$ is the only known fact, then α becomes a regularization parameter. In the last mentioned scheme, previous discussion tells that there is a regularization parameter α_{AD} for which the AD solution in its Lagrangian formulation is optimal, where the question of how to choose the parameter α is introduced as a new issue. In fact, system (30)-(31) can be rewritten as

$$(T_s^\star T_s + \alpha I)(\zeta, \psi) = T_s^\star(\tilde{\phi}, \tilde{\psi}), \quad T_s(\zeta, \psi) = \begin{pmatrix} A_s & B \\ 0 & I \end{pmatrix}\begin{pmatrix} \zeta \\ \psi \end{pmatrix}.$$

In other words, under the 2^{nd} assumption, the AD solution is the evaluation by Q^s of the first coordinate of the Tikhonov solution with parameter $\alpha = \alpha_K$ for the inverse problem $T_s\mathbf{x} = \mathbf{y}$. When the *a priori* information α_K becomes a regularization parameter α, it is important to recall that T_s is not a compact operator, so that one has to be carefully selecting the strategy to choose it. In some cases the strategy to choose the regularization parameter for Tikhonov solution, or the properties of its convergence rate to the exact solution depends on the property of compactness of the operator defining the inverse problem.

Remark 3.7. It is clear that ψ^δ converges to $\psi^\dagger$ when δ tends to 0, and because of that the pair $(\Phi^\delta, \psi^\delta)$ is a quasi solution of the inverse problem $T_0\mathbf{x} = \mathbf{y}$; however, the ill-posedness of the Cauchy problem is dominated by operator A, and this is why the approach $A\Phi = \rho$ has been chosen.

4. Dirichlet boundary data completion problem in cylindrical domains

Let Γ be a bounded domain in $\mathbb{R}^n$ $(n \ge 1)$, $\Omega = (0, a) \times \Gamma$, $\partial\Gamma$ the boundary of Γ, $\Sigma = [0, a] \times \partial\Gamma$, and $\Gamma_z = \{z\} \times \Gamma$, where z lies in $[0, a]$, Γ_a will take the place of Γ_1 in previous sections. The space $L^2(\Gamma_z)$ is isometric to $L^2(\Gamma)$ for all non negative real z, so we will consider all these spaces to be identical.

In this case, by the separation of variables method, solution of the problem (6)-(9) in Ω is given by

$$u(x; z) = \sum_{k=1}^{\infty} w_k(z) v_k(x), \quad w_k(z) = \frac{\Phi_k \cosh(z\lambda_k) + \frac{\psi_k}{\lambda_k}\sinh((a-z)\lambda_k)}{\cosh(a\lambda_k)}, \tag{35}$$

where $\{v_k\}_{k\in\mathbb{N}}$ is an orthonormal and complete system of $L^2(\Gamma)$ composed by eigenfunctions of minus Laplace operator defined on $H^2(\Gamma) \cap H_0^1(\Gamma)$, $\{\lambda_k\}_{k\in\mathbb{N}}$ is the set of corresponding eigenvalues. Eigenvalues are repeated according with their multiplicity, and their eigenspaces are finite dimensional. Let Φ_k and ψ_k be the k-th Fourier coefficient of Φ and ψ in $L^2(\Gamma)$ with respect to the system $\{v_k\}$. The analytic form of A, B, P and Q (as the inverse of P) are obtained from (35)

$$A\Phi = \sum_{k=1}^{\infty} \frac{\Phi_k}{\cosh(a\lambda_k)} v_k \ (A = W), \ B\psi = \sum_{k=1}^{\infty} \frac{\psi_k \sinh(a\lambda_k)}{\lambda_k \cosh(a\lambda_k)} v_k, \ P\Phi = \sum_{k=1}^{\infty} \lambda_k \frac{\Phi_k \sinh(a\lambda_k)}{\cosh(a\lambda_k)} v_k, \ Q\Psi = \sum_{k=1}^{\infty} \frac{\Psi_k \cosh(a\lambda_k)}{\lambda_k \sinh(a\lambda_k)} v_k. \tag{36}$$

The characterization of the admissible data set immediately follows from (12) and the analytic form of mappings A and B.

Theorem 4.1. *The pair $(\phi, \psi) \in L^2(\Gamma) \times L^2(\Gamma)$ is an admissible data if and only if*

$$\sum_{k=1}^{\infty} \lambda_k (\phi_k - \frac{\psi_k}{\lambda_k})^2 e^{2a\lambda_k} < \infty; \tag{37}$$

In that case, the function Φ on the left side of (12) *is defined by:*

$$\Phi = \sum_{k=1}^{\infty} \left[\phi_k \cosh(a\lambda_k) - \frac{\psi_k}{\lambda_k} \sinh(a\lambda_k) \right] v_k. \tag{38}$$

It is clear that recovering Φ from admissible data (ϕ, ψ) is severely ill-posed because singular values of A tend to zero exponentially. It is important to recall that $\lambda_1, \lambda_2, ...$ is a non decreasing sequence that goes to infinity. Notice that

$$-\ln^{-p}(WA)\zeta = \sum_{k=1}^{\infty} - \left(-2a\lambda_k - \ln\left(4(1 + e^{-2a\lambda_k})^{-2}\right)\right)^{-p} \zeta_k, \tag{39}$$

then, for $p = \frac{1}{2}$, and for K_a depending on K and a, $(\phi^\dagger, \psi^\dagger) \in M$ implies the logarithmic source condition:

$$\Phi^\dagger \in M_{p,K_a} = \left\{\Phi : \Phi = -\ln^{-p}(WA)\zeta, \ \|\zeta\| \le K_a\right\}. \tag{40}$$

The worst case error $w_A(\delta, K, \|\cdot\|_1)$ can be bounded by estimates in [24, 25].

On the other hand, the solution u must be in $H^1(\Omega)$, and $\{v_k\}$ is also an orthogonal set in $H_0^1(\Gamma)$ with inner product $(\nabla\cdot, \nabla\cdot)_{\mathrm{L}^2}$ satisfying $\|\nabla v_k\| = \lambda_k^2$. Hence, by a simple calculation from (35)

$$\langle \Phi, P\Phi \rangle_{E^{1/2} \times E^{-1/2}} = \sum_{k=1}^{\infty} \lambda_k \Phi_k^2 \frac{\sinh(a\lambda_k)}{\cosh(a\lambda_k)} v_k, \tag{41}$$

In the above equation, it is clear that $\|\nabla u\|$ is upper and lower bounded by a scalar multiple of $\left\|(-\Delta)^{1/4}\Phi\right\|$. In other words, a solution of the boundary data problem (1)-(4) exists if and only if $E^{1/2}(\Gamma)$ is the Hilbert space $H_{00}^{1/2}$ defined in [18] with inner product as in [2], all details can be reviewed at [16, chap 2]:

$$\langle f, g \rangle_{H_{00}^{1/2}} dS = \int_\Gamma (-\Delta)^{1/4} f (-\Delta)^{1/4} g dS \tag{42}$$

Then, $H_{00}^{1/2}(\Gamma)$ is also defined as the range of the compact operator $(-\Delta)^{-\frac{1}{4}}$, it follows that for every Φ in $H_{00}^{1/2}(\Gamma)$, there exists τ in $L^2(\Gamma)$ such that $(-\Delta)^{-\frac{1}{4}}\tau = \Phi$ i.e. operator Q in second assumption of *a priori* information in section 3.2 can be replaced by $(-\Delta)^{-\frac{1}{2}}$.

Remark 4.2. The sequence $\left\{\frac{\lambda_k}{\cosh(a\lambda)_k}\right\}$ is bounded (let us say by a constant C); then,

$$\|A\Phi\|_{E^{\frac{1}{2}}(\Gamma)} = \langle A\Phi, PA\Phi \rangle_{E^{1/2}(\Gamma) \times E^{-1/2}(\Gamma)} = \sum_{k=1}^{\infty} \frac{\lambda_k \Phi_k^2 \sinh(a\lambda_k)}{\cosh(a\lambda_k)^3} v_k \le C \|\Phi\|^2, \tag{43}$$

which means that A is continuous from $\mathrm{L}^2(\Gamma)$ to $E^{1/2}(\Gamma)$ (compact from $\mathrm{L}^2(\Gamma)$ to itself) i.e. we are in the case where A can be continuously extended: $\mathrm{D}(A) = \mathrm{L}^2(\Gamma)$. Hence, both choices for 2^{nd} assumption, ($(-\Delta)^{-\frac{1}{4}}$ or $Q^{\frac{1}{2}}$, are equivalents to the 1^{st} assumption, and produce an optimal regularized solution in this case, because the worst case error is considered in $\mathrm{L}^2(\Gamma_1)$. In further sections we will choose $(-\Delta)^{-\frac{1}{4}}$ instead of $Q^{\frac{1}{2}}$ for the regularization framework because the numerical example is set in a cylinder where $(-\Delta)^{-1/4}$ is easier to compute.

4.1. **Regularized solution in a semi-discretized scheme**

Let $V^{(m)}$ be the finite dimensional L^2 subspace generated by $\{v_1, v_2, \dots, v_m\}$, $V^{(m)\perp}$ the orthogonal complement of $V^{(m)}$, Pr the minimum distance projection over $V^{(m)}$, and $\mathrm{Pr}^{\perp}$ the corresponding minimum distance projection over $V^{(m)\perp}$. Since $\{v_1, v_2, \dots\}$ is a complete system of $\mathrm{L}^2(\Gamma)$, and still in the framework of the 2^{nd} assumption with $(-\Delta)^{-\frac{1}{4}}$ instead of Q^s with $s = \frac{1}{2}$ as mentioned at the end of the previous section: $\Phi^\dagger = (-\Delta)^{-\frac{1}{4}}\tau^\dagger$ with $\tau^\dagger$ in $\mathrm{L}^2(\Gamma)$, it holds true that $\left\|\mathrm{Pr}^{\perp}\,\Phi^\dagger\right\| = \sqrt{\sum_{k=m+1}^{\infty}\frac{\tau_k^2}{\lambda_k}} \le \delta$ whether $m = m_{\delta,K}$ is such that

$$\max\left\{\frac{K}{\lambda_{m_{\delta,K}}^{1/2}}, \left\|\mathrm{Pr}^{\perp}\tilde{\phi}\right\|, \left\|\mathrm{Pr}^{\perp}\tilde{\psi}\right\|\right\} \le \delta. \tag{44}$$

System $\{v_1, v_2, ...\}$ is complete and orthonormal in $\mathrm{L}^2(\Gamma)$, the analityc form of A and B in (4) shows that every v_k is an eigen function of both operators, as well. Hence, Pr commutes with A and B, implying that it also holds true that $A\,\mathrm{Pr}\,\Phi^\dagger = \mathrm{Pr}\,\phi^\dagger - B\,\mathrm{Pr}\,\psi^\dagger$, $\left\|\mathrm{Pr}\,\phi^\dagger - \mathrm{Pr}\,\tilde{\phi}\right\| \le \delta$, and $\left\|\mathrm{Pr}\,\psi^\dagger - \mathrm{Pr}\,\tilde{\psi}\right\| \le \delta$. This way, considering $(\mathrm{Pr}\,\tilde{\phi}, \mathrm{Pr}\,\tilde{\psi})$ as the measured Cauchy data instead of $(\tilde{\phi}, \tilde{\psi})$, and denoting the corresponding AD solution by $\Phi^{\delta, m_{\delta,K}}$ a new corollary of Theorem 3.3 is achieved.

Corollary 4.3. *The function $\Phi^{\delta, m_{\delta,K}}$ is an asymptotically optimal regularized solution to the inverse problem* (13) *whether* $(\mathrm{Pr}\,\tilde{\phi}, \mathrm{Pr}\,\tilde{\psi})$ *does not belong to* $\mathrm{cl}_{(\mathrm{L}^2(\Gamma_0))^2}(M)$.

Proof. Result follows immediately from previous discussion, and from Theorem 3.3:

$$\begin{aligned} \left\|\Phi^\dagger - \Phi^{\delta, m_{\delta,K}}\right\| &= \left\|\mathrm{Pr}\,\Phi^\dagger - \Phi^{\delta, m_{\delta,K}}\right\| + \left\|\mathrm{Pr}^{\perp}\,\Phi^\dagger\right\| \\ &\le w_A\left((1 + \|B\|)\delta_1, 2K\right) + \delta. \end{aligned} \tag{45}$$

□

The regularized solution $(\Phi^{\delta, m_{\delta,K}}, \psi^{\delta, m_{\delta,K}}) = (\Phi^{m,\delta,\alpha}, \psi^{m,\delta,\alpha})$ is given by solving the normal equations (46)-(47):

$$\left[A^2 + \alpha(-\Delta)^{1/4}\right]\Phi + AB\psi = A\,\mathrm{Pr}\,\tilde{\phi}, \tag{46}$$

$$BA\Phi + \left[B^2 + (1+\alpha)I\right]\psi = B\,\mathrm{Pr}\,\tilde{\phi} + \mathrm{Pr}\,\tilde{\psi}, \tag{47}$$

for $\alpha = \alpha_k$ such that $\left\|\left(\Phi^{m,\delta,\alpha}, \psi^{m,\delta,\alpha}\right)\right\|_0 = K$.

4.1.1. *Numerical example*

The scheme in subsection 4.1 is applied for $m = 550$ in the cylinder of revolution $\Omega = (0, a) \times \Gamma$, with $\Gamma = \{(x, y)|x^2 + y^2 = 1\}$. As (38) shows, the regularization error exponentially depends on how high the cylinder is (value of a) i.e the ill-posedness order of the inverse problem grows exponentially in function of cylinder height, making the problem virtually unavailable for larger values of a, so that, the example is developed for $a = 1$. In the present section the AD solution is compared with the Tikhonov regularization strategy, because the last one is still the most common method. The comparison is made in different regularity cases for exact data $\Phi^\dagger$:

- $\Phi^\dagger \in R((-\Delta)^{-\frac{1}{4}})$, where R means range. This is the poorest *a priori* information that can be provided to the operational formulation of the problem. This is a case of low regularity, where optimality for Tikhonov solution is not guaranteed ($\Phi^\dagger \in R((WA)^p(-\Delta)^{\frac{1}{4}})$, $p < 1$ with A as a L^2 continuous operator defined in $R((-\Delta)^{-\frac{1}{4}})$). Optimality for Tikhonov solution is considered with $X_1 = R((WA)^p)$, and $\|x\|_{X_1} = \|(WA)^{-p}x\|$in the worst case error definition in section 3; for a given $p \geq 0$. In this first case of low regularity: $p = 0$.
- $\Phi^\dagger \in R((WA)^p(-\Delta)^{-\frac{1}{4}})$, $1 \leq p \leq 2$. High regularity such that Tikhonov solution is optimal in the same sense as before. (see Theorem 2.12 [1, p. 38]).
- $\Phi^\dagger \in R((WA)^p(-\Delta)^{-\frac{1}{4}})$, $p > 2$. High regularity where no regularization parameter exists making optimal the Tikhonov solution ($\Phi^\dagger \in R((WA)^p)$, $p > 2$. In our particular case, $\|WA\| < 1$, by analogous proof of Theorem 1.21 in [1] one has that $\omega_A(\delta, K, \|\cdot\|_1) < O(\delta^{\frac{2}{3}})$, and Theorem 2.13, also in [1], tells us that Tikhonov solution cannot be optimal for $p > 2$.

For a given noise level $\delta > 0$, and regularization parameter α, the AD and Tikhonov solutions will be respectively denoted by $\Phi_{AD}^{m,\delta,\alpha}$, and $\Phi_T^{m,\delta,\alpha}$. The best possible regularized solutions in a synthetic example are defined by the smallest regularization parameters solving

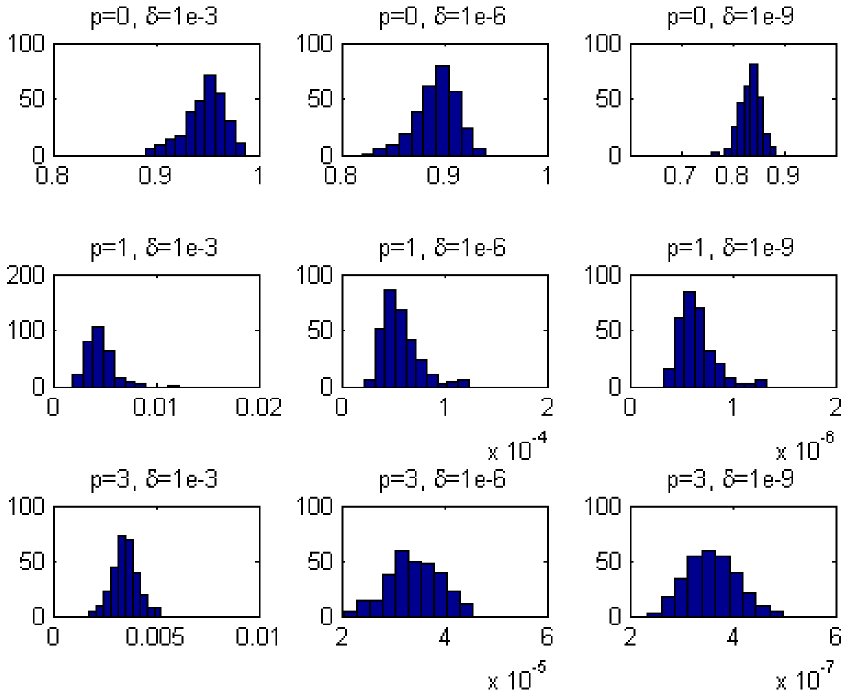
$$\alpha_{AD} = \operatorname{argmin}_{\alpha>0} \left\|\Phi^\dagger - \Phi_{AD}^{m,\delta,\alpha}\right\|, \qquad \alpha_T = \operatorname{argmin}_{\alpha>0} \left\|\Phi^\dagger - \Phi_T^{m,\delta,\alpha}\right\|.$$

The relative error of a regularized solution $\tilde{\Phi}$ is defined by $RE(\tilde{\Phi}) = \frac{\|\Phi^\dagger - \tilde{\Phi}\|}{\|\Phi^\dagger\|}$, as usual. Corresponding histograms are shown in Figure 1. Mean values and standard deviations (std) of a comparative numerical test between the best possible solutions for AD and Tikhonov solutions are presented in Table 1; the comparative is made for any $p \in \{0, 1, 3\}$ determining regularity of the exact solution ($\Phi^\dagger \in R\left((WA)^p\,(-\Delta)^{-1/4}\right)$), and noise level $\delta \in \{1e-3, 1e-6, 1e-9\}$. In all cases a 300 size sample of synthetic data $(\Phi^\dagger, \psi^\dagger, \phi, \psi)$ were built via the analytic form (35):

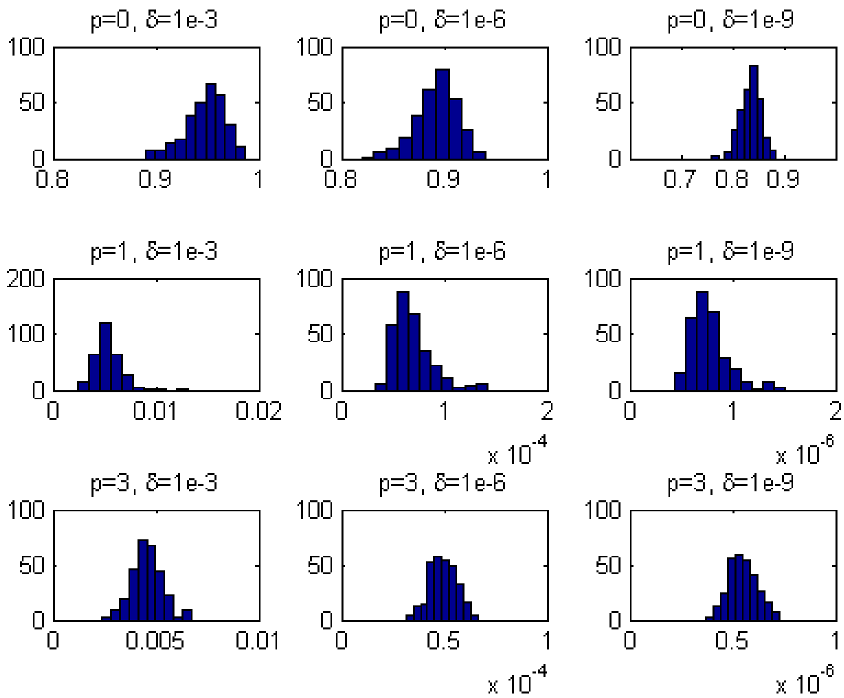
- $\zeta^\dagger, \psi^\dagger, \vec{\delta}^{(1)}, \vec{\delta}^{(2)} \in V^{(550)}$ such that any one of their Fourier coefficients in the $\{v_1, v_2...\}$ system is a pseudo random number with a normal standard distribution. Besides the error, we build a large sample of pseudo-random exact pairs $(\zeta^\dagger, \psi^\dagger)$ in $\mathrm{L}^2(\Gamma) \times \mathrm{L}^2(\Gamma)$ to control the regularity on $\Phi^\dagger$ through its Fourier series.
- $\Phi^\dagger = (WA)^p(-\Delta)^{-\frac{1}{4}}\zeta^\dagger$, according with the cases described above.
- $\tilde{\phi} = A\Phi^\dagger + B\psi^\dagger + \delta\frac{\vec{\delta}^{(1)}}{\|\vec{\delta}^{(1)}\|}$, and $\tilde{\psi} = \psi^\dagger + \delta\frac{\vec{\delta}^{(2)}}{\|\vec{\delta}^{(2)}\|}$.

$\Phi^\dagger \in$	δ	$RE(\Phi_{AD}^{n,\delta,\alpha_{AD}})$ mean value	std	$RE(\Phi_T^{n,\delta,\alpha_T})$ mean value	std
$R((WA)^0(-\Delta)^{-1/4})$	1E-3	0.947458241	0.019608435	0.947495676	0.019603288
	1E-6	0.892655869	0.019817953	0.892685598	0.019811939
	1E-9	0.835546278	0.020035893	0.835555588	0.02003477
$R((WA)(-\Delta)^{-1/4})$	1E-3	0.004486134	0.000219996	0.005334435	0.0013084
	1E-6	5.70165E-05	1.76458E-5	6.89085E-5	1.86605E-5
	1E-9	6.43584E-7	1.70329E-7	7.71689E-7	1.83486E-7
$R((WA)^3(-\Delta)^{-1/4})$	1E-3	0.003488645	0.000192304	0.004563972	0.000762609
	1E-6	3.41411E-5	5.07517E-6	4.95877E-5	6.81315E-6
	1E-9	3.59424E-7	5.00185E-08	5.49233E-7	6.91165E-8

Table 1. Mean values and standard deviations for the best possible solutions for AD and Tikhonov solutions. Samples size:300. Height of the cylinder: $a = 1$. For $p = 0$ the identity $I = (WA)^0$ is considered in $L^2(\Gamma_a)$.



(a) $RE_{AD}(\Phi)$



(b) $RE_T(\Phi)$

FIGURE 1. Histograms of $RE_{AD-T}(\tilde{\phi}, \tilde{\psi})$, in 300 size samples, with noise level error δ, and regularity of the exact data determined by p ($\Phi^\dagger \in R((WA)^p(-\Delta)^{-1/4})$). Height of the cylinder: $a = 1$. For $p = 0$ the identity $I = (WA)^0$ is considered in $L^2(\Gamma_a)$.

Figure 2 shows that low regularity on the exact solution implies a higher order on the worst case error. Two cases are seen, in the first one the exact solution is not regular enough to belong to a class of functions for which Tikhonov strategy behaves well, while the second example was made applying WA to the exact data in the first example, in order to include an example where best possible Tikhonov solution must be optimal. In both cases the shared Neumann component is the following non differentiable function given in polar coordinates:

$$\psi^\dagger(r,\theta) = \sqrt{|r-\theta|}(r\sin(3r)\cos(\theta))^3,$$

as long as Dirichlet boundary conditions are

$$\Phi^\dagger(r,\theta) = \frac{(1-r)\cos(\theta)^3}{r}, \text{ and } \quad \Phi^\dagger(r,\theta) = WA\frac{(1-r)\cos(\theta)^3}{r}; \quad \phi^\dagger = A\Phi^\dagger + B\psi^\dagger.$$

Cauchy data $(\phi^\dagger, \psi^\dagger)$ were perturbed in the same pseudo-random way than before with a noise level $\delta = 10^{-4}$ to obtain $(\tilde\phi, \tilde\psi)$.

Remark 4.4. In this case, an eigenfunction of $-\Delta$ has the form

$$v(r,\theta) = J_i(\lambda r)\cos(j\theta), \quad v(r,\theta) = J_i(\lambda r)\sin(j\theta), \tag{48}$$

where J_i is the special Bessel function of first kind and order i, and λ are such that $J_i(\lambda) = 0$.

4.2. **Link with the invariant embedding**

The boundary data completion problem in Γ_a has been approached as the first kind linear inverse problem in eq. (12). That can also be done in the framework of the invariant embedding in [2], with the slightly difference that the embedding is taken from a to 0 instead of from 0 to a, and state u defined by (6)-(9) is the only one used. The boundary data problem (6)-(9) is embedded in a family of boundary data problem defined in the sub-domains $\Omega_s = (s,a)\times\Gamma$. Consider the following families of BVP:

$$\begin{aligned} \Delta u_1^{(s)} &= 0 \text{ in } \Omega_s, \\ u_1^{(s)}\Big|_\Sigma &= 0, \\ u_1^{(s)}\Big|_{\Gamma_a} &= \Phi, \\ \frac{\partial u_1^{(s)}}{\partial\nu}\bigg|_{\Gamma_s} &= 0, \end{aligned} \qquad\qquad \begin{aligned} \Delta u_2^{(s)} &= 0 \text{ in } \Omega_s, \\ u_2^{(s)}\Big|_\Sigma &= 0, \\ u_2^{(s)}\Big|_{\Gamma_a} &= 0, \\ \frac{\partial u_2^{(s)}}{\partial\nu}\bigg|_{\Gamma_s} &= \zeta. \end{aligned}$$

Let $A(s) : H_{00}^{1/2}(\Gamma_a) \to H_{00}^{1/2}(\Gamma_s)$ be the Dirichlet to Dirichlet operator defined by: $A(s)\Phi = u_1^{(s)}\Big|_{\Gamma_s}$, and $B(s) : (H_{00}^{1/2}(\Gamma_s))' \to H_{00}^{1/2}(\Gamma_s)$ the Neumann to Dirichlet operator defined by $B(s)\zeta = u_2^{(s)}\Big|_{\Gamma_s}$. This way, equation (49) holds, and (12) is obtained from it, at $s = 0$:

$$u(s;\cdot) = B(s)\left.\frac{\partial u_2}{\partial\nu}\right|_{\Gamma_z} + A_a(s)\Phi. \tag{49}$$

Through the same formal derivation scheme as in [2, 9, 16], with $h = a - z$, all the following equations are fulfilled for $h \in (0,a)$

$$\tfrac{d}{dh}A(h) + B(h)\Delta_\Gamma A(h) = 0, \quad A(0) = I,$$

$$\tfrac{d}{dh}B(h) + B(h)\Delta_\Gamma B(h) = I, \quad B(0) = 0.$$

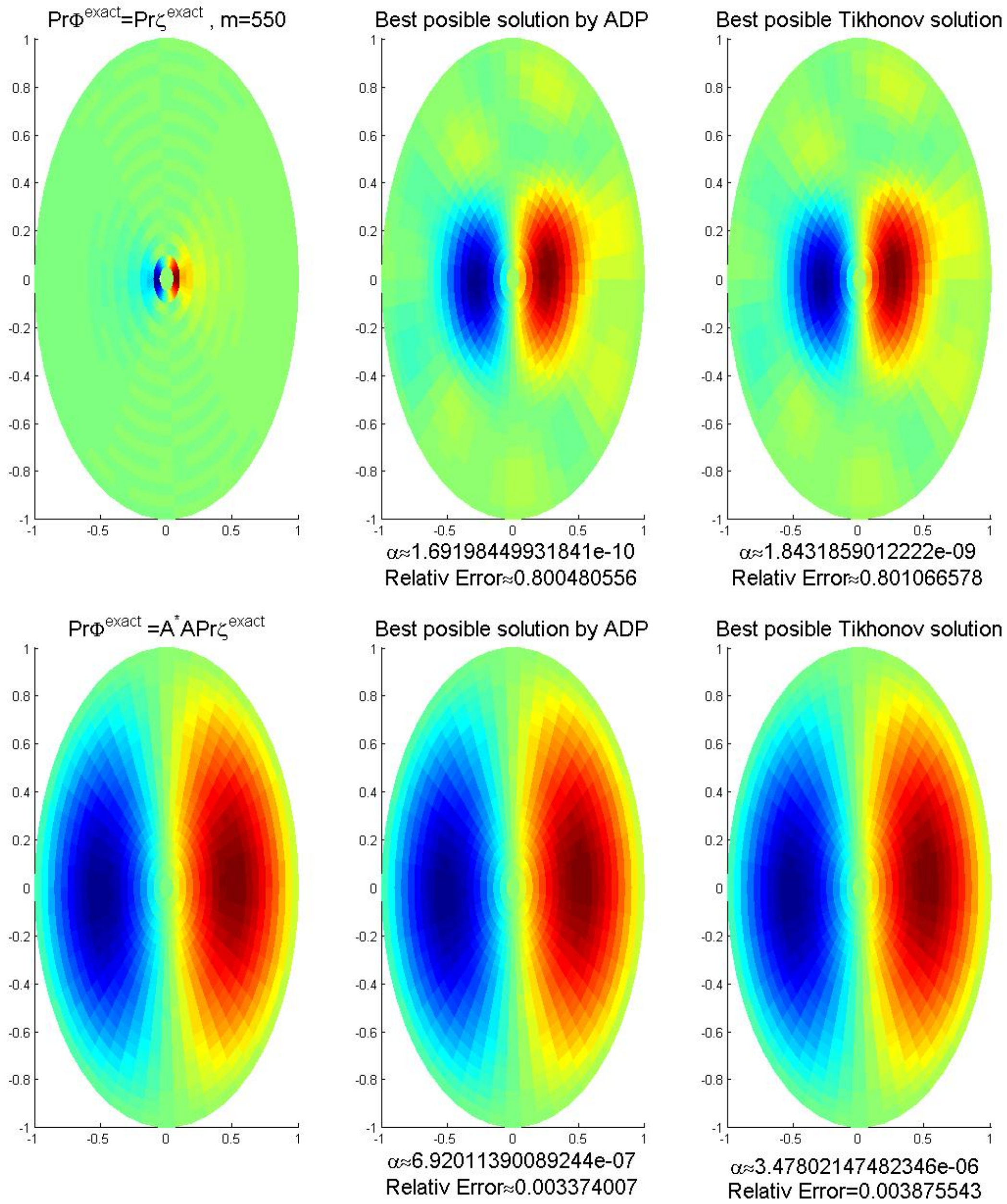


FIGURE 2. Comparison of AD and Tikhonov solutions. First line: $\Phi^{exact} = \Phi^{\dagger}$ is given by $((1-r)\cos(\theta)^3)/r$. Projection of Φ^{exact} over $V^{(550)}$, $Pr\Phi^{exact}$ (left), best possible AD solution (center), best possible Tikhonov solution (right). Second line: $\Phi^{exact} = WA\left(((1-r)\cos(\theta)^3)/r\right)$. Projection of Φ^{exact} over $V^{(550)}$, $Pr\Phi^{exact}$ (left), best possible AD solution (center), best possible Tikhonov solution (right). Semi-discretization scheme in section 4.1. Height of the cylinder: $a = 1$.

Now, operators $P(s) : H_{00}^{1/2}(\Gamma_a) \to (H_{00}^{1/2}(\Gamma_a))'$ and $W(s)(: H_{00}^{1/2}(\Gamma_s))' \to (H_{00}^{1/2}(\Gamma_a))'$ will be as in [16]: $\Omega^s = (0, s) \times \Gamma$, $P(s)\Phi = \left.\frac{\partial U_1^{(s)}}{\partial \nu}\right|_{\Gamma_a}$, and $W(s)\psi = -\left.\frac{\partial U_2^{(s)}}{\partial \nu}\right|_{\Gamma_a}$, where:

$$\begin{aligned} \Delta U_1^{(s)} &= 0 \text{ in } \Omega^s, & \Delta U_2^{(s)} &= 0 \text{ in } \Omega^s, \\ \left.U_1^{(s)}\right|_{\Sigma} &= 0, & \left.U_2^{(s)}\right|_{\Sigma} &= 0, \\ \left.U_1^{(s)}\right|_{\Gamma_s} &= \tau, & \left.U_2^{(s)}\right|_{\Gamma_s} &= 0, \\ \left.\frac{\partial U_1^{(s)}}{\partial \nu}\right|_{\Gamma_0} &= 0, & \left.\frac{\partial U_2^{(s)}}{\partial \nu}\right|_{\Gamma_0} &= \psi. \end{aligned}$$

According with [2, 9, 16]:

$$\tfrac{d}{ds}P + P^2 = -\Delta_\Gamma, \quad P(0) = 0,$$

$$\tfrac{d}{ds}W + PW = 0, \qquad W(0) = I.$$

Hence, the invariant embedding in [16] allows to set the normal equations of the AD solution depending on operators satisfying the above formal differential equations, also suggesting that it can be implemented in the same discretized and semi-discretized schemes than the factorization method in more complex geometries of Ω, or when the goal is to solve as fast as possible the same inverse problem for a set of measured Cauchy data recorded in a period of time, computing once for all the involved operator and avoiding time consuming numerical methods. The key of those statements is that the same regularization approach developed in sections 2 to 3.1 apply for the following cases:

- Ω a cylinder as before, where null Dirichlet boundary condition (7) is replaced by the Neumann one $\frac{\partial u}{\partial \nu}\big|_\Sigma = 0$, and the spaces E_0 and E_{00} are defined as follows

$$E_0 = \left\{ v \in H^1(\Omega) : \left.\frac{\partial u}{\partial \nu}\right|_{\Sigma} = 0 \right\}, \; E_{00} = \left\{ v \in E_0 : v|_{\Gamma_1} = 0 \right\},$$

- $\Omega = \Omega_0 \setminus \mathrm{cl}_{\mathbb{R}^n}(\Omega_1)$, where Ω_0 and Ω_1 are bounded domains in $\mathbb{R}^n$ with regular boundaries, and such that $\mathrm{cl}_{\mathbb{R}^n}(\Omega_1) \subset \Omega_0$. In this case $\Gamma_0 = \partial\Omega_0$, $\Gamma_1 = \partial\Omega_1$, boundary condition (7) is omitted, and

$$E_0 = E_{00} = \left\{ v \in H^1(\Omega) : v|_{\Gamma_1} = 0 \right\}.$$

In both of previous modifications of the inverse problem, $\mathrm{L}^2(\Gamma_1)^\perp = \left\{ \Phi \in \mathrm{L}^2(\Gamma_1) : \int_{\Gamma_1} \Phi dS = 0 \right\}$ must be the domain of Q. Both examples are developed in [16] in the framework o the factorization method when there exists a C^1 diffeomorphism between Γ_0 and Γ_1.

5. Discussion

The proposed solution was built following a new regularization scheme, which do not provide a general strategy in the regularization theory; however, it allows to take full advantage on all available *a priori* information about the solution for a given problem, following an intuitive way to build optimal approximations to the solution. When the measured Cauchy data is projected over the set of admissible data, we are looking for the nearest Cauchy data in a subclass of functions defined by strong enough *a priori* information. This subclass is such that the weak solution of the Cauchy problem is well defined on it, in other words, the approximation is optimal because we forced it.

The AD solution in this case leads to a constrained optimization problem, were constraints are defined by the available *a priori* information. If this information guarantees that the solution is a regular point for the given constraints, they can become regularization parameters via the equivalent Lagrangian multipliers. In that case and by the following modification $T_s(\Phi, \psi) = (A_s\Phi + B\psi, \psi)$ described in section 3, the AD solution can be thought as the image of a compact operator applied over the first component of the Tikhonov solution for the linear and non compact operator. Another way of looking at it is through eq. (34), where one can easily see that $(A_s^* A_S + \alpha I) A_s^* (\tilde{\phi} - B\psi^{\delta,\alpha})$ is the Tikhonov solution for the original inverse problem $A\phi = \rho$ were the perturbed output data have been continuously modified by eq. (33). Anyhow, it seems that careful modifications of well known strategies can be applied under the 2^{nd} assumption of *a priori* information, strategies like discrepancy principle, L$-$curve [15], U-curve, balancing principle [23], or ADP technique [12].

For more complex geometries of Ω, when the goal is to solve as fast as possible the same inverse problem for different output data recorded in a period of time, section 4.2 and [10,16] suggest that the invariant embedding can be an option for numerical implementations of the admissible data methodology.

The last remarkable aspect is that, applying this procedure, a kind of deterministic hypothesis test is given when δ and $\left\|(\Phi^\dagger, \psi^\dagger)\right\|_0$ are known. In practice, if the projection $(\phi^\delta, \psi^\delta)$ satisfies the inequality $\left\|(\phi^\delta, \psi^\delta) - (\tilde{\phi}, \tilde{\psi})\right\| \geq \sqrt{2}\delta$, then it is possible to say that there is something wrong with the device which recorded the Cauchy data, or, in the worst of the cases, that the model of the phenomena is not good enough for the desired purposes.

Finally, the obtained regularized solution was compared with Tikhonov regularization strategy, since it is still the most common applied regularization strategy, even though it is well known that it cannot be optimal for high enough smoothness of the input data in linear inverse problems [1, Theorem 2.13 p.39].

6. Conclusion

In cylindrical domains is enough to known the value of the natural norm of the exact Cauchy data $(\phi^\dagger, \psi^\dagger)$ in the class of admissible data; in the general assuming slightly stronger *a priori* information is enough. With the employed methodology, an optimal regularized approximation to the boundary data completion problem is attained. However, in the Lagrangian formulation, the regularized solution depends on the *a priori* information $K = \left\|(\Phi^\dagger, \psi^\dagger)\right\|_0$, which is rarely available in practice. The remaining research issue is to obtain a method to choose the parameter α in (28) that only depends on the measurement.

Numerical examples show that the best possible AD solution behaves well in a semi-discretized scheme; however, it is important to explore other discretization schemes for different, and more complex geometries of Ω, where computation of eigenfunctions v_k, and eigenvalues λ_k is a problem itself.

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